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# A STAGGERED MESH COMPACT DIFFERENCE SCHEME AND A PRESSURE–POISSON–EQUATION THAT SATISFIES THE EQUIVALENCY[1]

YU Xin
( *Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080* )

## Abstract

(1) A staggered mesh compact difference scheme is presented for solving the unsteady viscous incompressible Navier–Stokes equations. It is fourth order accurate both in the spatial direction and in the time direction, at least third order accurate on the boundary; (2) Describe a pressure–Poisson–equation that is equivalent to the discrete continuity equation provided the discrete momentum equations remain. The discrete continuity equation may have derivative boundary conditions, e.g., the compact difference scheme; (3) A new ADI iterative method is proposed. The pressure–Poisson–equation is in the discrete form. It is difficult to be solved with a usual ADI method. We translate it to be tridiagonal in each spatial direction of each step of the ADI iterations, then add a pseudo time term to get a tridiagonal equation which is easily to be solved; (4) The driven flow in a square cavity with $Re = 10000$ is simulated numerically.

## 1. Introduction

With the developing of the electronic computers, more and more physical problems can be simulated numerically. While there are still a lot of nonlinear problems need too much CPU time and too large computer memory to calculate. The improvement of the numerical methods can reduce the computer time and memory greatly, e.g., for a two dimensional unsteady problem, in order to reduce the error to $N^{-4}$, a second order accurate scheme needs about $(N^2)^3$ spatial and time grids, while a fourth order accurate scheme needs only $N^3$ grids. The ratio of them is $N^3$. The ratio for the computer time is even more. $N^3 = 4096$ when $N = 16$, $N^3 \approx 1.678 \times 10^7$ when $N = 256$.

The advantages of compact difference schemes over traditional methods include the relatively high order of accuracy using a compact stencil, a better (linear) stability, a better resolution for high frequency wave,and usually fewer boundary points to handle.[8,1,2,3,5,6,10]

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With a three points stencil in each time level, the scheme in this paper achieves fourth order accurate both in the spatial direction (§2.2) and in the time direction (§2.4), at least third order accurate on the boundary (§2.3). This paper is a revised version of [9].

## 2. A Fourth Order Accurate Staggered Mesh Compact Difference Scheme

The compact difference scheme in this paper improves in the treatment of the pressure boundary differences, and the discretization of the time derivative in [1][2].

**2.1 The Unsteady Viscous Incompressible Navier–Stokes Equations**:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{A}(\boldsymbol{V}) + \boldsymbol{\nabla} p = \boldsymbol{0}, \quad \text{in} \quad \Omega, \quad (\text{ where } \boldsymbol{A}(\boldsymbol{V}) = (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} - \nu\nabla^2\boldsymbol{V} \ ), \tag{2.1}$$

$$\text{div } \boldsymbol{V} = 0, \qquad \text{in} \quad \Omega \tag{2.2}$$

**2.2 A Two Dimensional Staggered Mesh Compact Difference Scheme**

The compact difference scheme described in this section has the fourth order accuracy for the spatial discretization of (2.1)(2.2), the first order accuracy for the time derivative in (2.1). We consider the discrete form of (2.1)(2.2)

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^*}{\Delta t} + \boldsymbol{\nabla}_h p^{n+1} = 0, \qquad (\text{ here } \boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n) \ ), \tag{2.3}$$

$$\text{div}_h \ \boldsymbol{V}^{n+1} = 0, \tag{2.4}$$

where $\boldsymbol{V}^n = (u^n, v^n)$, $\boldsymbol{A}(\boldsymbol{V}) = (uu_x + vu_y - \nu(u_{xx} + u_{yy}), uv_x + vv_y - \nu(v_{xx} + v_{yy}))$,$\boldsymbol{A}_h$, $\boldsymbol{\nabla}_h$, $\text{div}_h$ are discrete forms of $\boldsymbol{A}$, $\boldsymbol{\nabla}$, div respectively:

(1) $u_x, u_y$ in $\boldsymbol{A}_h(\boldsymbol{V}^n)(v_x, v_y$ in $\boldsymbol{A}_h(\boldsymbol{V}^n)$ similar):

$$\frac{u'_{i-1,j+\frac{1}{2}} + 4u'_{i,j+\frac{1}{2}} + u'_{i+1,j+\frac{1}{2}}}{6} = \frac{u_{i+1,j+\frac{1}{2}} - u_{i-1,j+\frac{1}{2}}}{2\Delta x}, \tag*{(2.5)$_1$}$$

$$\frac{u'_{i,j-\frac{1}{2}} + 4u'_{i,j+\frac{1}{2}} + u'_{i,j+\frac{3}{2}}}{6} = \frac{u_{i,j+\frac{3}{2}} - u_{i,j-\frac{1}{2}}}{2\Delta y}, \tag*{(2.5)$_2$}$$

(2) $u_{xx}, u_{yy}$ in $\boldsymbol{A}_h(\boldsymbol{V}^n)(v_{xx}, v_{yy}$ in $\boldsymbol{A}_h(\boldsymbol{V}^n)$ similar):

$$\frac{u''_{i-1,j+\frac{1}{2}} + 10u''_{i,j+\frac{1}{2}} + u''_{i+1,j+\frac{1}{2}}}{12} = \frac{u_{i+1,j+\frac{1}{2}} - 2u_{i,j+\frac{1}{2}} + u_{i-1,j+\frac{1}{2}}}{(\Delta x)^2}. \tag*{(2.6)$_1$}$$

$$\frac{u''_{i,j-\frac{1}{2}} + 10u''_{i,j+\frac{1}{2}} + u''_{i,j+\frac{3}{2}}}{12} = \frac{u_{i,j-\frac{1}{2}} - 2u_{i,j+\frac{1}{2}} + u_{i,j+\frac{3}{2}}}{(\Delta y)^2}. \tag*{(2.6)$_2$}$$

(3) $p_x$ in $\boldsymbol{\nabla}_h p$ ( $p_y$ similar):

$$\frac{p'_{i-1,j+\frac{1}{2}} + 22p'_{i,j+\frac{1}{2}} + p'_{i+1,j+\frac{1}{2}}}{24} = \frac{p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x}, \tag{2.7}$$

(4) $u_x$ in $\mathrm{div}_h \boldsymbol{V}$ ( $v_y$ in $\mathrm{div}_h \boldsymbol{V}$ similar):

$$\frac{u'_{i-\frac{1}{2},j+\frac{1}{2}} + 22u'_{i+\frac{1}{2},j+\frac{1}{2}} + u'_{i+\frac{3}{2},j+\frac{1}{2}}}{24} = \frac{u_{i+1,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}}{\Delta x}, \tag{2.8}$$

(5) $u$ in $uu_x$ in ( $\boldsymbol{V} \cdot \boldsymbol{\nabla}$ )$\boldsymbol{V}$: $u_{i,j+\frac{1}{2}}$, $v$ in $vv_y$ : $v_{i+\frac{1}{2},j}$, $u$ in $uv_x$:

$$u_{i+\frac{1}{2},j} = \frac{u_{i,j-\frac{1}{2}} + u_{i+1,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}} + u_{i+1,j+\frac{1}{2}}}{4} - \frac{1}{8}[(\Delta x)^2 u_{xx} + (\Delta y)^2 u_{yy}]_{i+\frac{1}{2},j}, \tag{2.9}$$

where $u_{xx}, u_{yy}$ can be get from the results of (2.6), $v$ in $vu_y$ similar,

(6) for $\boldsymbol{V}_t$: (2.3) uses the first order accurate difference. A fourth order accurate Runge–Kutta algorithm is given in §2.4.

### 2.3 Difference Formulae on the Boundary

Set $\Omega = \{(x,y)|0 < x < 1, 0 < y < 1\}, \Delta x = \Delta y = 1/N$. Consider the boundary condition $\boldsymbol{V}|_\Gamma = \boldsymbol{V}_\Gamma = (u^\Gamma, v^\Gamma)$, where $\Gamma = \partial\Omega$. We describe the difference formulae on the left edge $x = 0$.

$(1)_1$ $u_x|_{x=0} = -v_y^\Gamma$, (since (2.2)), (for $v$ : $v_y|_{y=0} = -u_x^\Gamma$) (2.10)

$(1)_2$ $v_x$ at $x = \frac{3}{4}\Delta x$ and $u_y$ at $y = \frac{3}{4}\Delta y$:

$$\frac{3v'_{\frac{1}{2},j} + v'_{\frac{3}{2},j}}{4} = \frac{v_{\frac{3}{2},j} - v_{0,j}}{\frac{3}{2}\Delta x}, \qquad \frac{3u'_{i,\frac{1}{2}} + u'_{i,\frac{3}{2}}}{4} = \frac{u_{i,\frac{3}{2}} - u_{i,0}}{\frac{3}{2}\Delta y}, \tag{2.11}$$

$(2)_1$ $u_{xx}$ at $x = \frac{10}{9}\Delta x$ ( $v_{yy}$ at $y = \frac{10}{9}\Delta y$ similar):

$$\frac{8u''_{1,j+\frac{1}{2}} + u''_{2,j+\frac{1}{2}}}{9} = \frac{9u_{0,j+\frac{1}{2}} - 16u_{1,j+\frac{1}{2}} + 7u_{2,j+\frac{1}{2}}}{6(\Delta x)^2} + \frac{u'_{0,j+\frac{1}{2}}}{3\Delta x}, \tag{2.12}$$

where $u'_{0,j+\frac{1}{2}} = -(v_y^\Gamma)_{0,j+\frac{1}{2}}$, cf (2.10),

$(2)_2$ $v_{xx}$ at $x = \frac{2}{3}\Delta x$ ( $u_{yy}$ at $y = \frac{2}{3}\Delta y$ similar):

$$\frac{5v''_{\frac{1}{2},j} + v''_{\frac{3}{2},j}}{6} = \frac{4}{3} \cdot \frac{2v_{0,j} - 3v_{\frac{1}{2},j} + v_{\frac{3}{2},j}}{\Delta x^2}, \tag{2.13}$$

(3) $p_x$ at $x = \Delta x$ ( $p_y$ at $y = \Delta y$ similar):

$$p'_{1,j+\frac{1}{2}} + \frac{p'_{1,j+\frac{1}{2}} - 2p'_{2,j+\frac{1}{2}} + p'_{3,j+\frac{1}{2}}}{24} = \frac{p_{\frac{3}{2},j+\frac{1}{2}} - p_{\frac{1}{2},j+\frac{1}{2}}}{\Delta x}, \tag{2.14}$$

(2.14) improves the only second order accurate difference formula in [1] [2] to third order accurate,

(4) $u_x$ in $\mathrm{div}\boldsymbol{V}$ at $x = \frac{1}{2}\Delta x$ ( $v_y$ at $y = \frac{1}{2}\Delta y$ similar):

$$\frac{2u'_{0,j+\frac{1}{2}} + 15u'_{\frac{1}{2},j+\frac{1}{2}} + u'_{\frac{3}{2},j+\frac{1}{2}}}{18} = \frac{u_{1,j+\frac{1}{2}} - u_{0,j+\frac{1}{2}}}{\Delta x}, \text{(where } u'_{0,j+\frac{1}{2}} = -(v_y^\Gamma)_{0,j+\frac{1}{2}} \text{)}, \tag{2.15}$$

(5) boundary values of $u_{xx}, u_{yy}$ in (2.9):

$$u_{xx}|_{x=\frac{1}{2}\Delta x} = \frac{u_x|_{x=\Delta x} - u_x|_{x=0}}{\Delta x}, \qquad u_{yy}|_{x=0} = u_{yy}^\Gamma|_{x=0}, \tag{2.16}$$

where $u_x|_{x=\Delta x}$ can be get from the results of (2.5), $u_x|_{x=0} = -v_y^\Gamma|_{x=0}$ since (2.2).

**2.4 The Fourth Order Accurate Runge–Kutta Method for the Time Derivative**

Define $\boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p$, where $p = p(\boldsymbol{V})$ satisfies $\mathrm{div}_h(\boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p) = 0$ . Thus $\boldsymbol{f}$ is a function of $\boldsymbol{V}$. To solve $\boldsymbol{V}_t + \boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{0}$, we use the fourth order accurate Runge–Kutta method:

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \frac{\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4}{6} = \boldsymbol{0}, \tag{2.17}$$

where $\boldsymbol{k}_1 = \boldsymbol{f}(\boldsymbol{V}^n)$, $\boldsymbol{k}_2 = \boldsymbol{f}(\boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{k}_1), \boldsymbol{k}_3 = \boldsymbol{f}(\boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{k}_2), \boldsymbol{k}_4 = \boldsymbol{f}(\boldsymbol{V}^n - \Delta t \boldsymbol{k}_3)$.

## 3. A Pressure–Poisson–Equation that Satisfies the Equivalency [1]

Let us consider the following discrete scheme,(e.g., the compact difference scheme in §2, and the second order accurate MAC scheme[4]),

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^*}{\Delta t} + \boldsymbol{\nabla}_h p^{n+1} = \boldsymbol{0}, \quad (\text{ where } \boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n) ), \tag{3.1}$$

$$\mathrm{div}_h\ \boldsymbol{V}^{n+1} = 0. \tag{3.2}$$

Define
$$\boldsymbol{NS}_h = \frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^*}{\Delta t} + \boldsymbol{\nabla}_h p^{n+1}, \tag*{(3.1)$_A$}$$

then (3.1) can be written to $\boldsymbol{NS}_h$=$\boldsymbol{0}$.

To avoid $\mathrm{div}_h(\boldsymbol{NS}_h)$ invalid, suppose (3.1) is defined on the difference mesh of $\boldsymbol{V}$, e.g., for the staggered mesh scheme in §2:

$$\boldsymbol{NS}_h \sim ((NS_h)^1_{i,j+\frac{1}{2}}, (NS_h)^2_{i+\frac{1}{2},j}), \qquad \boldsymbol{V}_h^n \sim (u^n_{i,j+\frac{1}{2}}, v^n_{i+\frac{1}{2},j}).$$

We write $\mathrm{div}_h\boldsymbol{V}$ in the form containing the boundary conditions: $D_h(\boldsymbol{V}, \boldsymbol{V}_\Gamma, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma)$, where $\boldsymbol{V}$ is defined on the inner difference points, $\boldsymbol{V}_\Gamma = \boldsymbol{V}|_\Gamma, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma = \boldsymbol{\nabla}\boldsymbol{V}|_\Gamma$. (The compact scheme in §2 does not use all components of $\boldsymbol{V}$. It uses only the components which can be get from $\boldsymbol{V}_\Gamma$ and the continuity equation: $\boldsymbol{\nabla}(\boldsymbol{V}_\Gamma \cdot \boldsymbol{n})$ and $\boldsymbol{\nabla}(\boldsymbol{V}_\Gamma \cdot \boldsymbol{\tau}) \cdot \boldsymbol{\tau}$, here $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit normal and tangent vectors on $\Gamma$. the MAC scheme [4] does not use $\boldsymbol{\nabla}\boldsymbol{V}_\Gamma$, i.e., there is no derivative boundary condition in the continuity equation). Thus the discrete continuity equation (3.2) can be written to

$$D_h(\boldsymbol{V}^{n+1}, \boldsymbol{V}_\Gamma^{n+1}, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma^{n+1}) = 0. \tag*{(3.2)$'$}$$

From (3.1) (3.1)$_A$, hence $\boldsymbol{NS}_h = \boldsymbol{0}, \boldsymbol{V}^{n+1} - \alpha\boldsymbol{NS}_h = \boldsymbol{V}^{n+1}$, (where $\alpha$ is an arbitrary constant). Therefore from (3.2)$'$ we get

$$D_h(\boldsymbol{V}^{n+1} - \alpha\boldsymbol{NS}_h, \boldsymbol{V}_\Gamma^{n+1}, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma^{n+1}) = 0. \tag{3.3}$$

**Theorem 3.1** For any arbitrary constant $\alpha$, (3.3)(3.1) are equivalent to (3.2)(3.1).

Set $\alpha = \Delta t$. Then (3.3) with (3.1) leads to the pressure–Poisson–equation

$$D_h(\boldsymbol{\nabla}_h p^{n+1}, \mathbf{0}, \mathbf{0}) = \frac{1}{\Delta t} D_h(\boldsymbol{V}^*, \boldsymbol{V}_\Gamma^{n+1}, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma^{n+1}) \tag{3.4}$$

provided $D_h(\boldsymbol{V}, \boldsymbol{V}_\Gamma, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma)$ is a linear function of $(\boldsymbol{V}, \boldsymbol{V}_\Gamma, \boldsymbol{\nabla}\boldsymbol{V}_\Gamma)$. RHS of (3.4) is $\frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}^*$ (cf (2.8)(2.15)) with $\boldsymbol{V}^*|_\Gamma = \boldsymbol{V}_\Gamma^{n+1}$, $(\boldsymbol{\nabla}\boldsymbol{V}^*)_\Gamma = \boldsymbol{\nabla}\boldsymbol{V}_\Gamma^{n+1}$ (see $u'_{0,j+\frac{1}{2}}$ in (2.15)). LHS of (3.4) is $\mathrm{div}_h(\boldsymbol{\nabla}_h p^{n+1})$ (see (2.7)(2.14) (2.8)(2.15)) with $(\boldsymbol{\nabla} p^{n+1})_\Gamma = \mathbf{0}$, (see $u_{0,j+\frac{1}{2}}$ in (2.15)), ( $\boldsymbol{\nabla}(\boldsymbol{\nabla} p^{n+1}))_\Gamma = \mathbf{0}$, (see $u'_{0,j+\frac{1}{2}}$ in (2.15)). These **numerical boundary conditions** do not affect the result of $p^{n+1}$ by theorem 3.1. See [4] §6.3.1.

So we can solve the pressure–Poisson–equation (3.4) for $p^{n+1}$, i.e., use (3.4)(3.1) or (3.3)(3.1) instead of (3.1)(3.2) to solve the Navier–Stokes equations (2.1)(2.2).

## 4. A New ADI Iterative Method for Solving the Pressure–Poisson–Equation

It is difficult to solve the discrete pressure–Poisson–equation (3.4) with a usual ADI method. We translate it to be tridiagonal in each spatial direction of each step of the ADI iterations ((4.5)–(4.7)), then add a pseudo time term to get a tridiagonal equation which is easily to be solved (see (4.7)), (a usual ADI method adds the pseudo time term at the beginning). (3.4) can be written to

$$\mathrm{div}_h(\boldsymbol{\nabla}_h p) = \frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}^*, \qquad (+\text{ boundary condition}) \tag{4.1}$$

The new ADI method in the $x$ direction (similar in the $y$ direction):

(1) calculate $p_y$ in $\boldsymbol{\nabla}_h p$ using the initial $p$ or the last step $p$, and the formulae corresponding to (2.7) (2.14):

$$\frac{(p_y)_{i+\frac{1}{2},j-1} + 22(p_y)_{i+\frac{1}{2},j} + (p_y)_{i+\frac{1}{2},j+1}}{24} = \frac{p_{i+\frac{1}{2},j+\frac{1}{2}} - p_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y}, (j = 2, 3, \cdots), \tag*{$(4.2)_1$}$$

$$(p_y)_{i+\frac{1}{2},1} + \frac{(p_y)_{i+\frac{1}{2},1} - 2(p_y)_{i+\frac{1}{2},2} + (p_y)_{i+\frac{1}{2},3}}{24} = \frac{p_{i+\frac{1}{2},\frac{3}{2}} - p_{i+\frac{1}{2},\frac{1}{2}}}{\Delta y}, \tag*{$(4.2)_2$}$$

(2) calculate $p_{yy}$ in $\mathrm{div}_h(\boldsymbol{\nabla}_h p)$ using the formulae corresponding to (2.8) (2.15):

$$\frac{(p_{yy})_{i+\frac{1}{2},j-\frac{1}{2}} + 22(p_{yy})_{i+\frac{1}{2},j+\frac{1}{2}} + (p_{yy})_{i+\frac{1}{2},j+\frac{3}{2}}}{24} = \frac{(p_y)_{i+\frac{1}{2},j+1} - (p_y)_{i+\frac{1}{2},j}}{\Delta y}, (j = 1, 2, \cdots)$$

$$\frac{2(p_{yy})_{i+\frac{1}{2},0} + 15(p_{yy})_{i+\frac{1}{2},\frac{1}{2}} + (p_{yy})_{i+\frac{1}{2},\frac{3}{2}}}{18} = \frac{(p_y)_{i+\frac{1}{2},1} - (p_y)_{i+\frac{1}{2},0}}{\Delta y}, \tag{4.3}$$

$$(\,(p_y)_{i+\frac{1}{2},0} = 0,\ (p_{yy})_{i+\frac{1}{2},0} = 0\,)$$

(3) $\quad p_{xx} = \frac{1}{\Delta t}\mathrm{div}_h \boldsymbol{V}^* - p_{yy}$ (4.4)

(4) solve the equations corresponding to (2.8) (2.15) for $p_x$ with the recursion

–correction method:

$$\frac{(p_{xx})_{i-\frac{1}{2},j+\frac{1}{2}}+22(p_{xx})_{i+\frac{1}{2},j+\frac{1}{2}}+(p_{xx})_{i+\frac{3}{2},j+\frac{1}{2}}}{24}=\frac{(p_x)_{i+1,j+\frac{1}{2}}-(p_x)_{i,j+\frac{1}{2}}}{\Delta x},(j=1,2,\cdots)$$

$$\frac{2(p_{xx})_{0,j+\frac{1}{2}}+15(p_{xx})_{\frac{1}{2},j+\frac{1}{2}}+(p_{xx})_{\frac{3}{2},j+\frac{1}{2}}}{18}=\frac{(p_x)_{1,j+\frac{1}{2}}-(p_x)_{0,j+\frac{1}{2}}}{\Delta x}, \tag{4.5}$$

$$(\,(p_x)_{0,j+\frac{1}{2}}=0,(p_{xx})_{0,j+\frac{1}{2}}=0\,)$$

(5) calculate the LHS of (2.7) and (2.14):

$$(p_x^*)_{i,j+\frac{1}{2}}=\frac{(p_x)_{i-1,j+\frac{1}{2}}+22(p_x)_{i,j+\frac{1}{2}}+(p_x)_{i+1,j+\frac{3}{2}}}{24},(i=2,3,4,\cdots)$$

$$(p_x^*)_{1,j+\frac{1}{2}}=\frac{25(p_x)_{1,j+\frac{1}{2}}-2(p_x)_{2,j+\frac{1}{2}}+(p_x)_{3,j+\frac{1}{2}}}{24}, \tag{4.6}$$

(6) differencing (2.7) (2.14), adding a pseudo time term, the resulting equation on $p$ is tridiagonal as follows, solve it to get $p$

$$\frac{(p_x^*)_{i+1,j+\frac{1}{2}}-(p_x^*)_{i,j+\frac{1}{2}}}{\Delta x}=\frac{p_{i-\frac{1}{2},j+\frac{1}{2}}-2p_{i+\frac{1}{2},j+\frac{1}{2}}+p_{i+\frac{3}{2},j+\frac{1}{2}}}{(\Delta x)^2}-\frac{p_{i+\frac{1}{2},j+\frac{1}{2}}}{\Delta t_*},(i=1,2\cdots)$$

$$\frac{(p_x^*)_{1,j+\frac{1}{2}}}{\Delta x}=\frac{p_{\frac{3}{2},j+\frac{1}{2}}-p_{\frac{1}{2},j+\frac{1}{2}}}{(\Delta x)^2}-\frac{p_{\frac{1}{2},j+\frac{1}{2}}}{\Delta t_{*1}}, \tag{4.7}$$

The corresponding formulae on the right and the upper boundaries can be obtained by translating $(4.2)_2,(4.3)_2,(4.5)_2,(4.6)_2,(4.7)_2$ by:

$$i\to N-i,\Delta x\to-\Delta x \text{ and } j\to N-j,\Delta y\to-\Delta y.$$

The computations in this paper and in [1] were performed by the above ADI method.

## 5. Computations of the Driven Flow in a Square Cavity

Consider the Navier–Stokes equations (2.1) (2.2) on $0<x<1$, $0<y<1$ with boundary conditions $\boldsymbol{V}=(-1,0)$, (on $y$=1), $\boldsymbol{V}$=(0,0), (on $y=0$, $x=0$, $x=1$), $Re=1/\nu$ =10000. We simulate the driven flow in a square cavity with the staggered mesh compact difference scheme with $\Delta x=\Delta y=1/N$, where $N=256$.

To verify the equivalency (theorem 3.1) further, in [1] we used a coarse grid ($N=8$), ADI iterations, reduced the error to $|\text{div}_h\boldsymbol{V}|<10^{-15}$ in the calculation. Here we adopt a multigrid method, $N=256, Re=10000$, reduce the error to $|\text{div}_h\boldsymbol{V}|<6\times10^{-14}$, This shows the pressure–Poisson–equation (4.1) (or (3.4)) can replace the continuity equation $\text{div}_h\boldsymbol{V}=0$ (suppose retaining the momentum equation (3.1)).

The streamlines (in figure 1) are contours of the stream function $\psi$. In order to obtain $\psi$, here we solve a Poisson equation deduced from

$$\frac{\psi_{i,j+1}-\psi_{i,j}}{\Delta y}=u_{i,j+\frac{1}{2}}+\frac{(\Delta y)^2}{24}u_{yy},\quad \frac{\psi_{i+1,j}-\psi_{i,j}}{\Delta x}=-(v_{i+\frac{1}{2},j}+\frac{(\Delta x)^2}{24}v_{xx}), \tag{5.1}$$

where $u_{yy}, v_{xx}$ are at least second order accurate, (then (5.2) with (5.1) is a fourth order accurate approximation of the continuity equation (2.2),

$$[(\psi_{i+1,j+1} - \psi_{i+1,j}) - (\psi_{i,j+1} - \psi_{i,j})] - [(\psi_{i+1,j+1} - \psi_{i,j+1}) - (\psi_{i+1,j} - \psi_{i,j})] = 0. \quad (5.2)$$

As a result, (2.3)–(2.7),(5.1),(5.2),(2.9)–(2.14),(2.16) is another fourth order accurate compact difference scheme for solving the Navier–Stokes equations (2.1)(2.2)).

$\|\cdot\|$ in figure 3 is defined by

$$\|\boldsymbol{V}\| = \frac{1}{2N}\left(\sqrt{\sum_{i,j}(u_{i,j+\frac{1}{2}})^2} + \sqrt{\sum_{i,j}(v_{i+\frac{1}{2},j})^2}\right), \qquad (\boldsymbol{V} = (u,v)), \quad (5.3)$$

$$\boldsymbol{V}_4 = \boldsymbol{V}|_{t=t_1+536.655}, \quad \boldsymbol{V}_5 = \boldsymbol{V}|_{t=t_1+542.340}.$$

The computing time of the results in [2] was relatively short. Here we employ the multigrid method for computation of the pressure. Longer calculation shows that the time period of the numerical solution is about $T = 6.727$, approximating $T = 6.36$ (the period given in [3]). It can be seen that the numerical solution is still not completely periodic. In figure 3 near $t = t_1 + 549$

$$\min_n \|(\boldsymbol{V}|_{t=t_1+n\Delta t}) - \boldsymbol{V}_5\| = \|(\boldsymbol{V}|_{t=t_1+n_1\Delta t}) - \boldsymbol{V}_5\| \approx 1.13 \times 10^{-5}.$$

With careful calculations we can get

$$\min_{0\le n\le 100} \|(\boldsymbol{V}|_{t=t_1+n_1\Delta t+n\Delta t/100}) - \boldsymbol{V}_5\| \approx 7.15 \times 10^{-6}.$$

The results in [3] demonstrates that the period shown by the figures of the streamlines is a quarter of that shown by the figure of the velocity at the geometric center. [7] gave an analogous result. This coincides with the results in figure 3 and figure 2 here. From figure 2, $u(\frac{1}{2},\frac{1}{2},t)$ and $v(\frac{1}{2},\frac{1}{2},t)$ can be approximated by

$$|A(t)|\sin(2\pi t/T + \theta_1) + |B(t)|\sin(2\pi t/(T/4) + \theta_2) + C(t), \quad (5.4)$$

where $T$ also changes with time $t$, $|B|$ becomes positive in figure 2a, $|B| \approx |A|$ in figure 2b, $|B| \gg |A|$ in figure 2c.

## References


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